Gabor frames from Cauchy kernels

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joint with Yu. Belov, A. Kulikov.

I. Gabor analysis

$f \in L^2(\mathbb{R})$ $\qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\, e^{-2i\pi t\xi}\, dt$

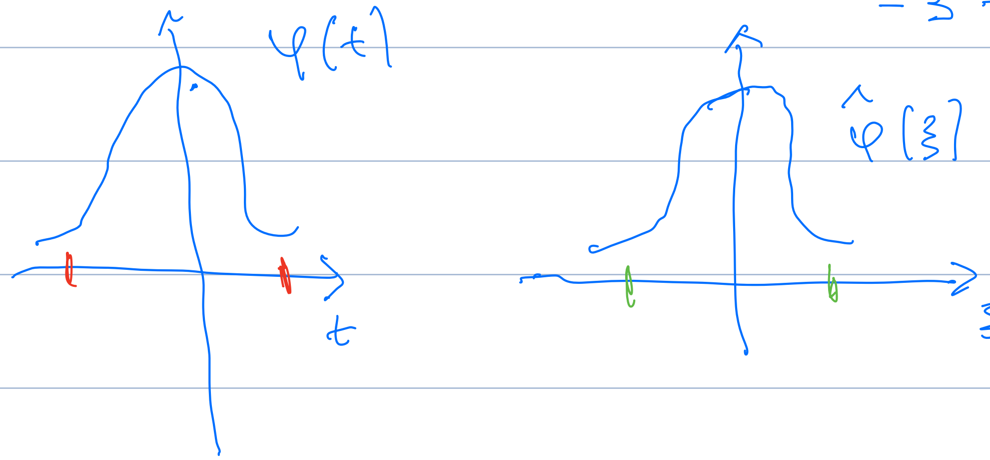
$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2i\pi t\xi}\, d\xi$$

Q: How one gets information both on frequency and location ?

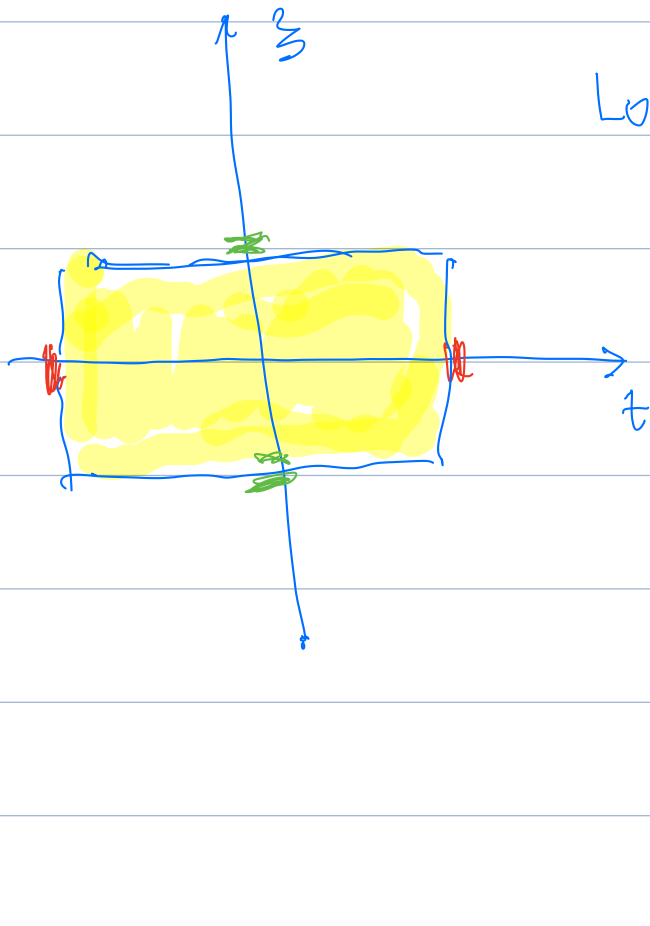
A: Take both Fourier and inverse Fourier transforms

(Another version: Short-time Fourier transform)

Example



Concentrated near zero both in time and frequency

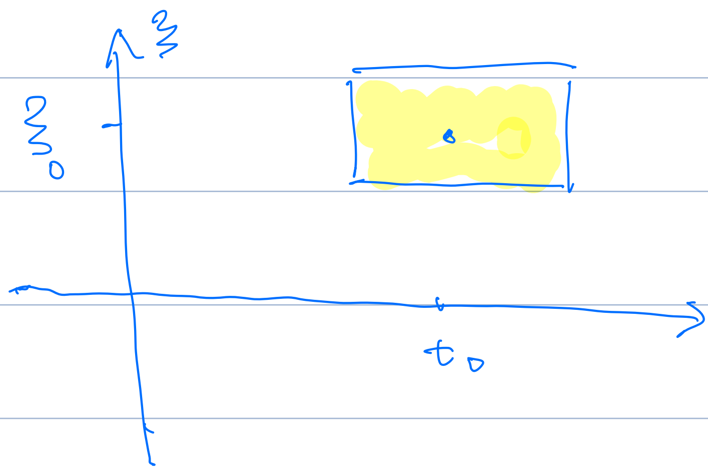


Location φ in time-frequency plane near $(0,0)$

$t_0 \in \mathbb{R}$, $\xi_0 \in \mathbb{R}$

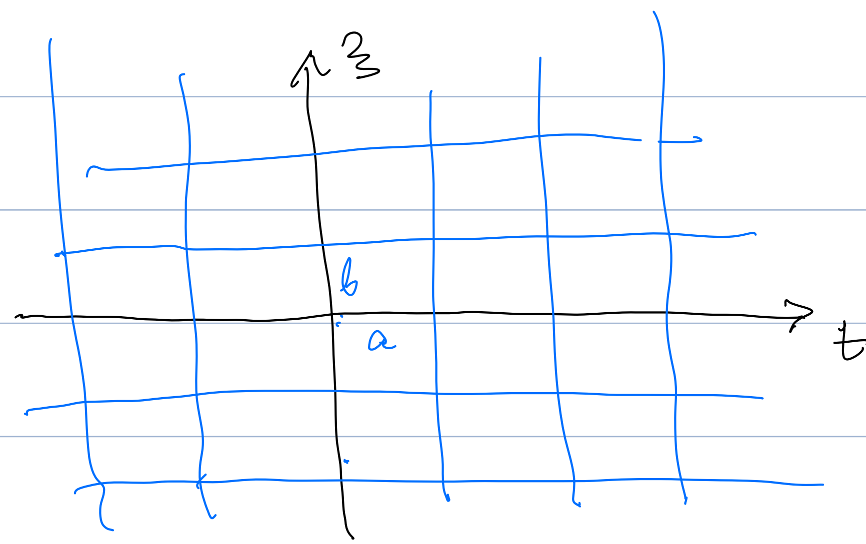
Time-frequency shift :

$$\varphi_{t_0,\xi_0}(t) = e^{-2i\pi\xi_0 t}\,\varphi(t-t_0)$$



Idea of Gabor analysis

Cover the whole time-frequency plane by shifts of "time-frequency supports" of φ and represent any $f \in L^2(\mathbb{R})$ as a sum of the corresponding shifts



Gabor representation:

$$f(t) = \sum_{m,n} c_{m,n} \, e^{-2i\pi n b t} \, \varphi(t - am)$$

Waltz in A Minor

Fryderyk Franciszek Chopin

B. 150

Frédéric François Chopin
(1810–1849)

Q: When this is possible?

$$f(t) = \sum c_{m,n} \, \varphi_{am,bn}(t)$$

1. Choice of φ
2. Choice of parameters a, b.
3. Can we replace the lattice by some irregular covering?

$$a\mathbb{Z} \times b\mathbb{Z} \longrightarrow \underline{\Lambda \times M} \longrightarrow \mathcal{L} = \{(\lambda, \mu)\} \subset \mathbb{R}^2$$

todays topic,

with $\varphi(t) = \dfrac{1}{t - i w}$

(Cauchy kernel.)

Notation: Gabor system

$\varphi \in L^2(\mathbb{R})$, $\mathcal{L} = \{(\lambda, \mu)\} \subset \mathbb{R}^2$

$$\mathcal{G}(\varphi, \mathcal{L}) = \{\varphi_{\lambda,\mu}(t)\}_{\lambda,\mu \in \mathcal{L}} = \{e^{-2i\pi\mu t} \varphi(t-\lambda)\}_{\lambda,\mu}$$

Definition: $\mathcal{G}(\varphi, \mathcal{L})$ is a frame if

$$A\|f\|^2 \leq \sum_{(\lambda,\mu)\in\mathcal{L}} |\langle f, \varphi_{\lambda,\mu}\rangle|^2 \leq B\|f\|^2$$

for any $f \in L^2(\mathbb{R})$,

Hints:

Think about Parseval inequalities or about stability

Basic fact:

$\mathcal{G}(\varphi, \mathcal{L})$ is a frame $\Rightarrow$

$\Rightarrow$ there is a Gabor representation

$$f(t) = \sum c_{\lambda,\mu} \varphi_{\lambda,\mu}(t)$$

$\Rightarrow$ Basic question

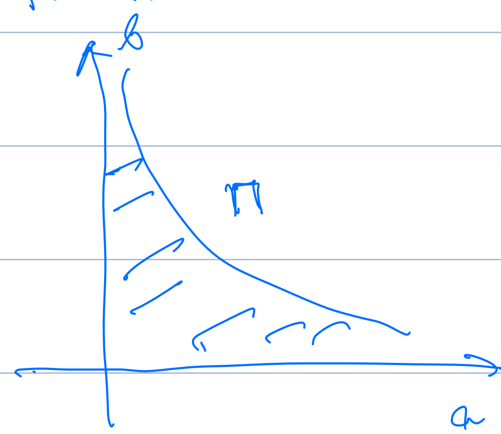
When $\mathcal{G}(\varphi, \mathcal{L})$ is a frame?

Common sense: $\mathcal{L}$ should be sufficiently dense.

Fact φ - slightly good, $a \cdot b \geq 1$ $\Longrightarrow$

area of the rectangle

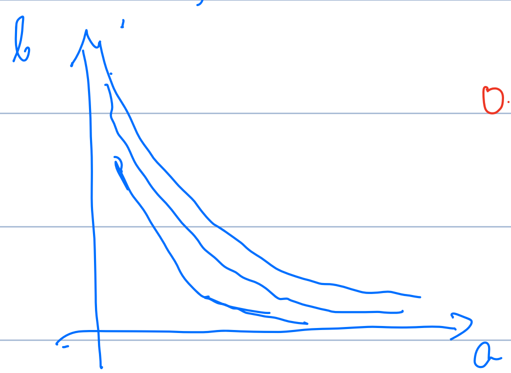
$\mathcal{G}(\varphi, a\mathbb{Z} \times b\mathbb{Z})$ is NOT a frame



History (mainly about rectangular lattices)

- all $(a,b) \in \Pi$
  - $\varphi(t) = e^{-\pi t^2}$ - Seip 92, Yu. L. 92.
  - $\varphi(t) = e^{-|t|}$, $(\cosh t)^{-1}$, $\chi_{(0,\infty)} e^{-t}$
    Jansen, Strohmer.
  - totally positive functions - Grochenig, Stokler.
  - some rational functions - Belov, Kulikov, Yu. L.

- Not all $(a,b) \in \Pi$,
  Yu. L, Ness: $\varphi(t) = t e^{-\pi t^2} \Rightarrow ab \neq \frac{n}{n+1}$



Open problem:
Is this the only exceptions?

- Janssen, Dai & Sun $\varphi(t) =$
- Dai and Zhu. $\varphi(t) =$

Our setting:

$$\varphi(t) = \frac{1}{t - i w}, \quad w > 0$$

$\Lambda \subset \mathbb{R}$, $M \subset \mathbb{R}$

When

$$g(\varphi; \Lambda \times M) = \left\{ \frac{e^{-2i\pi\mu t}}{t - \lambda - i w} \right\}_{\lambda \in \Lambda, \mu \in M}$$

is a frame?

<u>Definition:</u> $M = \{\mu_n\}_n \subset \mathbb{R}$, $\mu_n < \mu_{n+1}$.

M is locally finite if

- $\beta(M) := \sup\{\mu_{n+1} - \mu_n\} < \infty$

$$\beta_n := \mu_{n+1} - \mu_n$$

- $\sup_{x \in \mathbb{R}} \#(M \cap (x, x+1)) < \infty$, We need additional definition to describe Λ.

Definition:

1 Paley-Wiener space: $\beta > 0$

$$PW_{[0,\beta]} = \left\{ f(t) = \int_0^\beta e^{2i\pi t\xi}\,\hat{f}(\xi)\,d\xi,\ \hat{f} \in L^2[0,\beta) \right\}$$

Properties: $f(t)$ - entire function

$$f(t+is) = \begin{cases} o(1), & s \to +\infty \\ < Ce^{2\pi|s|\beta}, & s \to -\infty \end{cases}$$

In particular the Cauchy formula.

$$a > 0,\ \operatorname{Im} z > 0 \Rightarrow \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{f(t)\,e^{iat}}{t-z}\,dt = f(z)\,e^{iaz}$$

$$\operatorname{Im} z < 0 \Rightarrow \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{f(t)\,e^{iat}}{t-z}\,dt = 0$$

Definition $\Lambda = \{\lambda\} \subset \mathbb{R}$ sampling for $PW_{[0,\beta]}$ if

$$a\|f\|^2 \le \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \le b\|f\|^2$$

In terms of frames:

$\{e^{2i\pi\lambda t}\}_{\lambda\in\Lambda}$ is a frame in $L^2(0,\beta]$

Example: $\alpha \geq \beta$ – $\frac{1}{\alpha}\mathbb{Z}$ is sampling.

General description: Ortega, Seip.

Theorem

$\mathcal{G}\left(\frac{1}{t-i\omega}, \Lambda \times M\right)$ is a frame

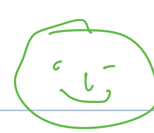
$\Longleftrightarrow$

M is locally finite and
Λ is sampling for $PW_{[0,\beta(M)]}$.

Idea of proof:

- Spectral decomposition of functions in L^2 according M.
- Cauchy formula for each component.
- Invertibility of the intertwining operator

Sounds sofisti-
cated



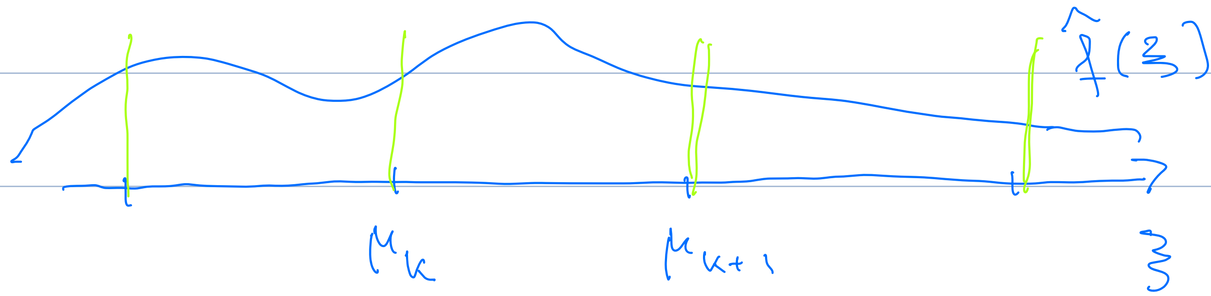
We need:

$$A\|f\|^2 \leq \sum_{\lambda, n} \Big| \Big\langle \underbrace{\frac{e^{-2i\pi\mu_n t}}{t-\lambda-i\omega}, \bar{f}}_{\text{notation} \to C_{\lambda,\mu_n}} \Big\rangle \Big|^2 \leq B\|f\|^2$$

$$c_{\lambda,\mu_n} = \int_{-\infty}^{\infty} \frac{e^{-2i\pi\mu_n t}}{t-\lambda-i\omega} f(t)\, dt.$$

This is the starting point.

1 Spectral decomposition



$$f_k(t) = \int_{\mu_k}^{\mu_{k+1}} \hat{f}(\xi)\, e^{2i\pi\xi t}\, d\xi$$

$$f(t) = \sum_k f_k(t)\,; \quad \|f\|^2 = \sum_k \|f_k\|^2$$

$$f_k(t) = e^{2i\pi\mu_k t} \underbrace{\int_0^{\mu_{k+1}-\mu_k} \hat{f}(\xi+\mu_k)\, e^{2i\pi\xi t}\, d\xi}_{h_k(t) \in PW_{[0,\beta(M)]}}$$

Respectively

$$c_{\lambda,\mu_n} = \sum_k \int_{-\infty}^{\infty} h_k(t)\, \frac{e^{2i\pi(\mu_k-\mu_n)t}}{t-\lambda-i\omega}\, dt$$

2. Cauchy formula

$$\int_{-\infty}^{\infty} h_k(t)\,\frac{e^{2i\pi(\mu_k-\mu_n)t}}{t-\lambda-i\varpi}\,dt =$$

$$= \begin{cases} h_k(\lambda+i\varpi)\,e^{2i\pi(\mu_k-\mu_n)(\lambda+i\varpi)}, & \mu_k \geq \mu_n \\ 0 & \mu_k < \mu_n \end{cases}$$

I.e.

$$c_{\lambda,\mu_n} = \sum_{k \geq n} h_k(\lambda+i\omega)\, e^{2i\pi\mu_k\lambda}\, e^{-2i\pi\mu_n\lambda}\, e^{-2i\pi(\mu_k-\mu_n)}$$

$$c_{\lambda,\mu_n} = \sum_{k\geq n} h_k(\lambda+i\omega)\, e^{2i\pi\mu_k\lambda}\, e^{-2i\pi\mu_n\lambda}\, e^{-2\pi(\mu_k-\mu_n)\omega}$$

$$d_{\lambda,\mu_n} := c_{\lambda,\mu_n} e^{2i\pi\lambda_n} \quad ; \quad \vec{d}_{\mu_n} = \{d_{\lambda,\mu_n}\}_\lambda$$

$$\|\{c_{\lambda,\mu_n}\}\|^2_{\ell^2(\Lambda\times M)} = \sum_n \|d_{\mu_n}\|^2$$

Respectively

$$\omega_{\lambda,\mu_k} = h_k(\lambda+i\omega)\, e^{2i\pi\lambda\mu_k} \quad ; \quad \vec{\omega}_\lambda = \{\omega_{\lambda,\mu_k}\}_k$$

$$\sum_\lambda \|\vec{\omega}_\lambda\|^2 = \sum_{\lambda,k} |h_k(\lambda+i\omega)|^2 \asymp \|f\|^2$$

because Λ is sampling.

3. Intertwining

$$\vec{\alpha}_{\mu_n} = A\vec{\omega},$$

This is the red part.

$$A = (a_{k,\mu_n}); \quad a_{k,\mu_n} = \begin{cases} e^{-2\pi(\mu_k - \mu_n)}, & k \geq n \\ 0, & k < n \end{cases}$$

We need to proof invertibility of A.

"Main relation"

$$k > n \Rightarrow \mu_k - \mu_n = (\mu_k - \mu_{k-1}) + \dots + (\mu_{n+1} - \mu_n)$$

<u>Notation</u> $\gamma_n = 2\pi\omega(\mu_{n+1} - \mu_n)$

$$B = (b_{p,q}) \qquad b_{p,q} = \begin{cases} e^{-\gamma_p} & q = p+1 \\ 0 & \text{otherwise.} \end{cases}$$

$$A = I + \sum_{j \geq 1} B^j \qquad (*)$$

For N large enough

$$\gamma_n + \cdots + \gamma_{n+N} \geq 1 \Rightarrow \|B^N\| \leq e^{-w}$$

$\Rightarrow$ Series $(*)$ converges to $(I - B)^{-1}$

Bingo!